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The Importance of Structural Integrity in Grain Design.

by
M. L. Williams

THE IMPORTANCE OF STRUCTURAL INTEGRITY ≠

IN GRAIN DESIGN

M. L. Williams*

INTRODUCTION

During the past several years solid propellant power units have changed in many ways. Notwithstanding the increase in specific impulse, probably the most striking of all is the remarkable growth in its physical size. This past and probably future growth has been accompanied by a proportionate rise in the cost of propellant units. The point of reviewing these facts is to stress an obvious but important fact. Grains are not cheap. The corollary is that it behooves the rocket engineer to bend every effort to design a grain which will not abort. He should carefully consider the use of scale models, laboratory testing and analytical procedures.

Turning specifically to the problem of structural design, we find that there is a fundamental yet simple distinction between the liquid and solid propellant fuel. The liquid fuel supports only hydrostatic compression; the solid fuel, on the other hand, not only withstands the same loading, but also varying amounts of tension and shear stress. From the mechanical standpoint this is a mixed blessing. If the grain material will take load, it is inefficient not to use this capability. But the material will also tend to absorb load, irrespective of the designers wishes. As a minimum requirement, therefore, a grain must be designed such that its proclivity to absorb load does not contribute to a rocket failure. Then, if possible, design it in such a way that its ability to absorb load contributes to a higher loading fraction, lower system weight, or improved overall performance. To reiterate, the structural integrity of a solid grain has a much stronger interaction with the overall system design than the liquid fuel, because a structural failure in the solid grain can contribute to off-design burning or even catastrophic ignition.

With these introductory remarks, it is desired to emphasize the economic and technical feasibility of conducting more sophisticated structural analyses of the grain than has heretofore been customary.

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≠Extracted from the Proc. 15th Meeting of the Joint Army-Navy-Air Force Solid Propellant Group, Washington D. C., June 1959.

HISTORICAL

Several years ago the Physical Properties Panel (SPIA) began to encourage some theoretical work in the area of viscoelasticity[1] pointed specifically toward eventual propellant analysis. Over the past five to seven years appreciable mathematical progress has been made, primarily through the effort of Lee[2] and his co-workers in extending the initial contributions of Alfrey[3], Tsien[4], Read[5], Eirich[6], and others. At the risk of some oversimplification, one may now say that if the grain material is linearly viscoelastic*, it is in principle possible to deduce the time dependent displacements, strains, and stresses, providing the solution to an analogous elastic (time independent) problem is known.

ENGINEERING ANALYSIS

More or less coincident with these developments in viscoelastic theory[7], there were several analyses directed primarily toward engineering applications. Among the first of these was the common design rule for the minimum acceptable strain based upon the square of the external to internal radius ratio, $(b/a)^2$. This factor originated from the maximum elastic hoop strain at the inner portion of the web. The exact relation for a long grain of constant wall thickness subjected to internal pressure and completely enclosed in a very rigid case is[8]

$$\epsilon_\theta = \left.\frac{u}{r}\right|_{r=a} = \frac{(1+\nu)(1-2\nu)}{E}\left[\frac{(b/a)^2-1}{1+(1-2\nu)(b/a)^2}\right] \qquad (1)$$

The controlling geometric factor is thus seen to be the square of the external to internal radius ratio. When multiplied by some empirical value to account for strain concentrations at internal star points and subsequent yielding, an ultimate strain value was generated which a laboratory propellant sample was required to achieve in a simple tensile test at a strain rate simulating ignition.

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*If a time dependent stress produces an associated time dependent strain, then if doubling the magnitude of the stress holding the mode shape of the time variation the same also doubles the strain magnitude without changing the shape of its time dependence, the response is said to be linearly viscoelastic.

For some time this design rule proved adequate, especially for the smaller grains which were designed mainly upon a build-and-test philosophy - and properly so, it might be added, considering the economics of the development. On the other hand, some designers began to suspect that improved criteria would soon be necessary. At this time viscoelastic theory was not in a particularly appropriate form for design application, and it was natural to seek some sort of compromise. Considering then that at the lower operating temperatures at least, the grain behaved nearly elastically, several analyses were made on this basis. One of the earliest directed specifically toward grain designs was that of Wise,[8] presented before this group 2 years ago. In that paper he analyzed the plane strain situation for an infinitely long grain with a rigid case subjected to internal pressure and uniform temperature rise. Upon choosing an ultimate strain criterion based on a uniaxial tensile test, he then predicted maximum operating pressures for tubular grains as a function of temperature. A comparison with experimental firing data showed quite reasonable agreement in the low temperature or elastic range. Wise also emphasized the necessity for improved material property data, particularly for Poisson's ratio, ν . In this connection note that (1) above is very sensitive to ν. This point is amply illustrated in Figures 2 and 3 taken from Wise. They show the hoop strain and propellant-case pressure, p' , as a function of the b/a ratio for an internal pressure, p_i . One may observe that for low b/a values, the case carries most of the load. It is evident that in addition to (i) a basic geometric factor $(b/a)^2$, and (ii) a form or stress concentration factor to account for departures from a tubular design, (iii) material properties, such as Young's modulus, E, and Poisson's ratio, as well as their rate dependence, will also play an important part in the grain behavior.

NON-TUBULAR CONFIGURATIONS

As indicated above, departure from a tubular geometry to a more realistic cross sectional configuration presents formidable analytical difficulties, although approximations can be made.[8] Durelli[9] at the Armour Research Foundation however proposed an experimental plane stress photoelastic set up by which he was able to deduce the maximum elastic strains and therefore determine the appropriate form factor in a given shape. He proceeded to demonstrate its feasibility by testing several shapes and thus determining their relative efficiency. Carrying the suggestion one step further, Ordahl and Williams[10] applied this technique to study the stress concentration factors for a systematic variation of the several design parameters for a series of internal burning slotted grains (Figure 4). This preliminary data was presented in convenient chart form (Figure 5) for easy use.

It can be seen that in addition to the previous factor b/a, which can be expressed alternately in terms of web fraction $w/b = 1-a/b = 1-(b/a)^{-1}$, one also has a slot or star radius parameter, w/ρ, number of slots, and some data on star angle. Thus, inasmuch as these factors were essentially based upon stress rise over that existing in a tubular grain, and also covered the bonded grain enclosed in an elastic, as opposed to a completely rigid case, it had been demonstrated that the designer had at his disposal a rapid and convenient method of analyzing the cross section of pressurized star grains in the elastic range.

At the Physical Properties Panel meeting last year, Williams[11] presented, as part of a survey paper, preliminary data demonstrating next the feasibility of obtaining steady state thermal stress concentrations for star grains subjected to a constant temperature difference across the web. This was again done photoelastically and incorporated a voltage analogy for determining the temperature distribution in the grain which may incidentally have further applications. This data, now completed[12] for the same configurations as in the pressure loading, was also presented in chart form (Figure 6) and referred to the tubular grain for which analytical solutions are available.

PROBLEM STATUS

The results discussed so far were to a large extent covered in the survey report[11] made to the Physical Properties Panel last year, wherein an attempt was made to assess the then current analysis methods and tools available to the analyst. In brief, it was concluded that there was much activity in measuring material properties, relatively little in stress analysis and practically none in failure theory. It did appear, however, that several approaches to the problem were becoming well defined, and that particularly in view of the larger sizes of current designs, it was timely - if not already belated - to mount a coordinated attack on the problem. Stated in another way, it seemed that certain developments in the theory of viscoelasticity were to the place where they could be applied directly to engineering problems, providing certain material property information was obtained and a failure theory established.

Based upon this hypothesis, an ad hoc Sub-Committee on Strain Analysis was formed to determine how well balanced the related fields of endeavor were and to make recommendations which would assist in achieving the objective outlined at the beginning of this presentation - to be able to evaluate the structural integrity of a grain early in the design stages with a minimum loss of development time and as little testing as possible.

Nearly all the previous remarks lie in the "water-over-the-dam" category, and it is desirable to summarize a few of the results which have been presented in more detail elsewhere, and to supplement another recent introductory treatment.[13] The first of these relates to analysis above the brittle or glassy modulus temperature T_g. Recall first that a large amount of emphasis was placed on elastic analysis, for two reasons. First, techniques are well known and second, it applied to low temperature operation. However, a third very important reason also exists when an earlier statement in this paper is recalled. Viscoelastic theory states that time dependent analysis can be carried out in principle of the (linear) stress-strain-time relation for the particular grain is known and if the analogous elastic problem has been solved. A collection of useful elastic solutions for pressure, temperature, and environmental bondings is included in Reference 14.

Hence, if we agree that designers can now determine, for example, the elastic stresses in the cross section of a case-bonded star grain subjected to internal pressure plus a temperature difference across the web, a way is now theoretically open to predict the time dependent behavior. To reiterate, this depends upon

1. A knowledge of the (linearly viscoelastic stress-strain relation which must be obtained experimentally.
2. A compatible mathematical description of the material, classically in terms of a spring (Hooke element), a dashpot (Newtonian element), or various series and parallel combinations of such elements,

plus the essential ingredient after the above <u>strain analysis</u> has been completed, namely

3. A theory or criterion for failure,

so the <u>strength analysis</u> can be carried out and the design job finished.

AN ILLUSTRATION

In order to fix our ideas, consider as a simple example the problem treated by Wise,[8] namely a long internally pressurized cylinder completely restrained at its outer periphery and at the ends by a material whose rigidity is very much greater than that of the propellant. The maximum elastic strain at the inside of the tubular grain (Figure 3) is, by (1),

$$\varepsilon_\theta = \frac{3p_i}{(3K+\mu)} \left[\frac{(b/a)^2 - 1}{1 + \frac{3\mu}{3K+\mu}\left(\frac{b}{a}\right)^2} \right] \tag{1a}$$

after a conversion to the shear modulus, $\mu = E/[2(1+\nu)]$, and the bulk modulus, $K = E/[3(1-2\nu)]$. The object now is to extend the analysis and determine the time dependent strain, assuming the grain material is linearly viscoelastic.

First of all, recall that there are essentially two types of deformation, (i) dilatation or volume change without distortion and (ii) distortion or shear accompanied by no change in volume of the element. The first is characterized by the bulk modulus and the second by the shear modulus. It is only natural, therefore, to expect that viscoelastic materials possess the same characteristic, but time dependent, features. It turns out that indeed this is true if the general three-dimensional stress-strain relation is cast in the form

$$\left[a_m \frac{\partial^m}{\partial t^m} + \cdots + a_0\right](\text{stress}) = \left[b_m \frac{\partial^n}{\partial t^n} + \cdots + b_0\right](\text{strain}) \tag{2}$$

or, in shorthand,

$$O_1[\sigma(x_k, t)] = O_2[\varepsilon(x_k, t)] \tag{2a}$$

Explicitly, if one employs the tensor notation,

$$\sigma_{ij} = \lambda \vartheta \delta_{ij} + 2\mu\,\varepsilon_{ij} \tag{3}$$

where σ_{ij} and ε_{ij} are stress and strain components respectively, where in terms of the displacements

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

and $\vartheta \equiv \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$. Furthermore, define the deviators

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \tag{4}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \tag{5}$$

which are the regular stress and strain components less the effect due to pure dilatation or hydrostatic pressure, then Lee[7] has shown that the linear viscoelastic stress-strain relations are completely characterized by

$$P[s_{ij}] = Q[e_{ij}] \tag{6}$$

$$R[\sigma_{ii}] = S[\varepsilon_{ii}] \tag{7}$$

where P, Q, R, and S are linear differential operators with constant coefficients, e.g.

$$Q = \sum_{n=0}^{q} q_n \frac{\partial^n}{\partial t^n} \quad , \quad \text{etc.} \tag{8}$$

similar to (2a) above. If one next uses the technique of Laplace transforms,

$$\bar{Q}(p) = \int_0^\infty e^{-pt} Q(t)\,dt$$

these operators can be converted to simple polynomials

$$\bar{Q}(p) = \sum_{1} q_r p^r \tag{9}$$

which, as they are independent of position coordinates x_K, permits one to write

$$\bar{s}_{ij}(x_K, p) = \frac{\bar{Q}(p)}{\bar{P}(p)} \bar{e}_{ij}(x_K, p) \tag{10}$$

$$\bar{\sigma}_{ii}(x_K, p) = \frac{\bar{S}(p)}{\bar{R}(p)} \bar{\varepsilon}_{ii}(x_K, p) \tag{11}$$

Compare now the conventional elastic stress-strain relations, also employing the deviators. These are, directly from (3), using (4) and (5),

$$s_{ij}(x_K) = (2\mu) e_{ij}(x_K) \tag{12}$$

$$\sigma_{ii}(x_K) = (3K)\, \varepsilon_{ii}(x_K) \tag{13}$$

and the formal basis of the similarity between the viscoelastic and elastic situation is immediately evident. Referring to (10), (11), the transformed viscoelastic stresses and strains can be interpreted as having analogous quantities, namely the ratios $\bar{Q}/\bar{P}$, $\bar{S}/\bar{R}$, as the elastic stresses and strains, i.e. the shear and bulk moduli respectively. To emphasize this relation, define

$$\frac{\bar{Q}(p)}{\bar{P}(p)} \equiv 2\mu \tag{14}$$

$$\frac{\bar{S}(p)}{\bar{R}(p)} \equiv 3K \tag{15}$$

THE VISCOELASTIC PROBLEM

Within certain restrictions upon similar transformations of the equations of equilibrium and the boundary conditions not of concern here in demonstrating the principle, one first solves the elastic problem as in (1a), then identifies the strain, stress, and moduli with their appropriate transforms and writes, for the transformed radial strain at x_K (or in this case r) = a at the internal radius,

$$\bar{\varepsilon}_\theta(a,p) = \frac{3\,\bar{p}_i(p)}{2\left[\frac{\bar{S}(p)}{\bar{R}(p)} + \frac{1}{2}\frac{\bar{Q}(p)}{\bar{P}(p)}\right]} \cdot \frac{(b/a)^2 - 1}{\left[1 + \frac{3}{2}\,\frac{\frac{\bar{Q}(p)}{\bar{P}(p)}\cdot\left(\frac{b}{a}\right)^2}{\frac{\bar{S}(p)}{\bar{R}(p)} + \frac{1}{2}\frac{\bar{Q}(p)}{\bar{P}(p)}}\right]} \tag{16}$$

where $\bar{p}_i(p)$ represents the transform of the applied time dependent pressure variation $p_i(t)$.

Thus, we see that profiding the material property variation is known so that explicit formulas are available for $\bar{P}(p)$, $\bar{Q}(p)$, $\bar{R}(p)$, and $\bar{S}(p)$, the time dependent strain $\varepsilon_\theta(a,t)$ can be computed by taking the inverse Laplace transform of (16),

$$\varepsilon_\theta(a,t) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\bar{\varepsilon}_\theta(a,p)\,dp \tag{17}$$

As this step for the physical strain hinges upon the ability to represent the material, consider what would be done in a simple case.

In the elastic case, we have stress proportional to strain and hence the appropriate model would be a (Hooke) spring, i.e. force proportional to elongation. For creep, however, we expect rate effects and associate deformation with velocity or time rate of change of elongation, i.e. a viscous element or (Newtonian) dashpot. In the more general case than one would expect a combination of such elements to be needed in associating stress and strain, or mathematically expressing the ratio $\bar{Q}/\bar{P}$ and $\bar{S}/\bar{R}$. A summary of some of the more common elements along with their typical responses are given in Figure 7.[14]

Without going into the important detail as to how one plans and translates laboratory data into quantitative values for the appropriate spring and/or dashpot elements for a given propellant material, let us first assume that volume change is solely elastic. Mathematically speaking, this effect can be achieved by a spring (Figure 7a) such that using (11),

$$\frac{\bar{\sigma}_{ii}}{\bar{\varepsilon}_{ii}} = \frac{\bar{S}(p)}{\bar{R}(p)} = 3K \,, \qquad \text{i.e. a constant bulk modulus} \tag{18}$$

On the other hand, let us assume the distortion has a time dependence adequately represented by a spring and dashpot in series (Maxwell model, Figure 7d). Thus,

$$\frac{\bar{s}_{ij}}{\bar{e}_{ij}} = \frac{\bar{Q}(p)}{\bar{P}(p)} = \frac{p}{\frac{1}{m}p + \frac{1}{n}} = 2\mu \tag{19}$$

so that the transformed strain becomes

$$\bar{\varepsilon}_\theta(a,p) = \frac{3\bar{p}_i(p)}{2\left[3K + \frac{1}{2}\frac{p}{\frac{1}{m}p+\frac{1}{n}}\right]} \cdot \frac{(b/a)^2 - 1}{\left[1 + \frac{3}{2}\,\frac{\frac{p}{\frac{1}{m}p+\frac{1}{n}}\cdot\left(\frac{b}{a}\right)^2}{3K + \frac{1}{2}\frac{p}{\frac{1}{m}p+\frac{1}{n}}}\right]}$$

$$= \frac{3\bar{p}_i(p)\,(\lambda^2-1)}{m\left[1+3\lambda^2+(6K/m)\right]} \cdot \frac{p + (m/n)}{p + \left[(6Km/n)/(6K+(1+3\lambda^2)m)\right]} \tag{20}$$

which, after the inverse transformation gives, using for convenience a unit step pressure, and $\varepsilon_\theta(a, o) = 0$, i.e. zero initial strain,

$$\varepsilon_\theta(a,t) = \frac{p_i(\lambda^2-1)}{2K}\left\{1 - \left[1 - \frac{1}{1+\frac{(3\lambda^2+1)m}{6K}}\right]e^{-t/\tau}\right\} \tag{21}$$

where the time constant, τ, is

$$\tau = \frac{\eta}{m}\left(1 + \frac{(1+3\lambda^2)m}{6K}\right) ; \quad \lambda \equiv b/a \tag{22}$$

It may easily be shown that at $t \rightarrow 0$, the strain approaches its elastic value (1) which, of course, implies that the Maxwell dashpot element has not had time to relax, and the spring absorbs the entire load. On the other hand, for a long time compared to τ, the strain approaches a significantly larger limiting value

$$\varepsilon_\theta(a, t \text{ large}) \rightarrow \frac{p_i(\lambda^2-1)}{2K} = \frac{p_i}{E}\left[\frac{3}{2}(\lambda^2-1)(1-2\nu)\right] \tag{23}$$

controlled entirely by the elastic bulk response. Hence, one has the possibility of eventually exceeding a critical strain value even though the load was held instantaneously, or for short times.

Suppose a static pressure test is carried out so that the b/a ratio is unchanged as time increases. If b/a = 2, Figure 2 shows for $\nu = 1/4$ an initial strain value of 0.62Pi/E which may eventually reach over three and one-half times this value, 2.25Pi/E, depending upon the rate constant τ. In order to examine the potential time dependent failure situation, compute the strain rate being induced in the propellant. From (21),

$$R(a) = \frac{\partial \varepsilon_\theta(a,t)}{\partial t} = \frac{p_i(\lambda^2-1)}{2K}\left[1 - \frac{\eta}{m\tau}\right]\frac{e^{-t/\tau}}{\tau} \tag{24}$$

So that now we are in position to compute strain and strain rate in the propellant as a function of time. Indeed, it can be easily shown they are simply related

$$\varepsilon_\theta(a) = \frac{p_i(\lambda^2-1)}{2K} - \frac{R(a)}{\tau} \tag{25}$$

where the first term is recognized as the long term strain and $\varepsilon_\theta(a)$ has its elastic value as a lower limit.

FAILURE CRITERION

Now, if we knew the ultimate strain permissible in the propellant as a function of strain rate, we would be in position to compute a strain margin of safety as a function of time.

$$M.S.(t) = \frac{\varepsilon_\theta(a,t)}{\varepsilon_{\theta_{limit}}(a,t)} - 1 \tag{26}$$

Fortunately, Smith[15] has presented data to indicate that there appears to be an engineering correlation of these variables at least in uniaxial tension. In addition and as an extra dividend, it also provides for the temperature variable - a quantity held constant in this illustration.

Figure 8 shows a typical such curve taken from Reference 15 and based on constant strain rate data where a_T is the temperature shift factor developed from the work of Williams, Landel, and Ferry.[16]

$$a_T = \frac{-K_1(T-T_g)}{K_2+(T-T_g)} \tag{27}$$

T_g is the glassy modulus temperature and the constants K_1 and K_2 are empirical values determined for a particular composition.

Thus, at a given temperature, if the strain rate is known, the percent strain at maximum stress is found from the figure and the denominator quantity in (26) is determined and the structural analysis completed.

A FURTHER EXAMPLE

In the previous case, a static pressure and rigid boundary was assumed. During an actual firing, however, the value of b/a decreases and approaches unity at burnout. It is therefore possible that a judicious matching of burning rate against strain rate would permit one to maintain the strain at or below the initial value by selecting material properties, and hence, τ, such that a tendency of the strain to increase with τ is compensated by the faster burning down of the b/a ratio.

In this case, theoretical analysis again can be of value. Radok and Lee[17] for an elastically reinforced hollow cylinder, and Williams[18] for a sphere have analyzed the viscoelastic behavior for an internal boundary which is burning away. Again, without giving the details of the analysis, the maximum strain and corresponding strain rate can be calculated during the firing cycle, compared to the failure criterion and the structural integrity examined.

CONCLUSIONS

It is not the purpose of this paper to make a strain analysis of a grain shape, but rather to present some background which is hoped to indicate that quantitative analysis can be useful in the design of solid rocket motors. On the other hand, it is also appropriate to point out shortcomings in the present effort which results from a combination of several factors.

In the above illustrations, it was implicitly assumed that; (i) the applied internal pressure-time curve was known, (ii) the grain was rigidly restrained, although the extension to the elastic restraint condition is straight forward,[19] (iii) the stress-strain data not only existed and was linearly viscoelastic but could be represented mathematically or elastic in dilatation and isotropically Maxwell in distortion, (iv) the propellant was homogeneous, (v) the maximum hoop strain was the significant quantity, in contrast to distortional strain or some function of stress, (vi) the Smith master curve for ultimate properties not only applied to propellants rather than pure resins, but that (vii) the uniaxial tensile data for failure applied to failure in the biaxial strain field of the propellant grain.

Nevertheless, the possibility of a direct analysis procedure has been demonstrated. By implication, however, the two important practical limitations are:

1. Lack of the propellant material representation in appropriate mathematical form for the various materials commonly used in grains.
2. Positive knowledge of a failure criterion; specifically, whether the association of a one-dimensional test specimen with a triaxial grain failure is legitimate.

It is natural to expect that considerable effort would be expended in these two areas. Inasmuch as the summary of such work has been reported elsewhere[20] for the Strain Analysis Sub-Committee, let it suffice for the moment to confirm that various investigators are conducting analytical work in theoretical linear and non-linear visco-elasticity, approximate analysis techniques for problems that are intractable for exact solution, model representation, failure criteria utilizing both dilatation and shear test data for non-uniaxial stress fields, and some steady and transient thermal stress investigations including grain curing. Several particular problem areas have been recognized and in this discussion can only be itemized:

1. Environmental loadings - shock and vibration as well as thermal,
2. Fuller understanding of thermal cycling during cure or in-transit movement of the motors,
3. Slump during storage (long time) or in flight (short time),
4. Finite end effects and associated linear and bonding problems,
5. Continuing effort to understand the macroscopic behavior of propellant materials.

In addition, and as mentioned at the outset, there are other problems related to the analysis methods of complete motor tubes which are very important, namely head-cylinder casing problems particularly in conjunction with liner analysis and the thin high strength cases, nozzle thermal stress analysis, and resulting design synthesis. However, it is felt to be appropriate for the time being to concentrate upon the grain itself. Eventually, if the situation warrants, the related problems will be considered.

In conclusion, I should like to express a personal opinion that there is a shortage in the companies of experimentalists and analysts whose efforts are directed specifically toward the structural analysis of grains rather than the necessary areas of resolving immediate crash problems and establishing quality control. A proper appreciation by company management and a concerted effort by engineering analysts is necessary in order to provide an opportunity for assessing quantitatively the importance of the structural integrity contribution to the best system design of the solid propellant motor.

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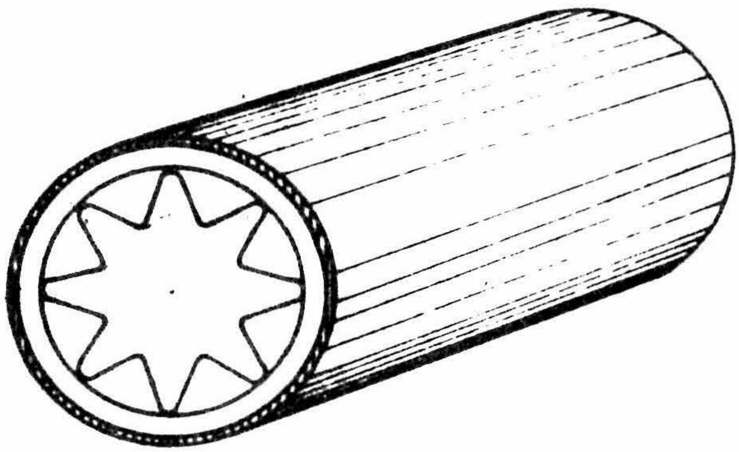


Fig. 1. Illustration of an internal burning star grain.

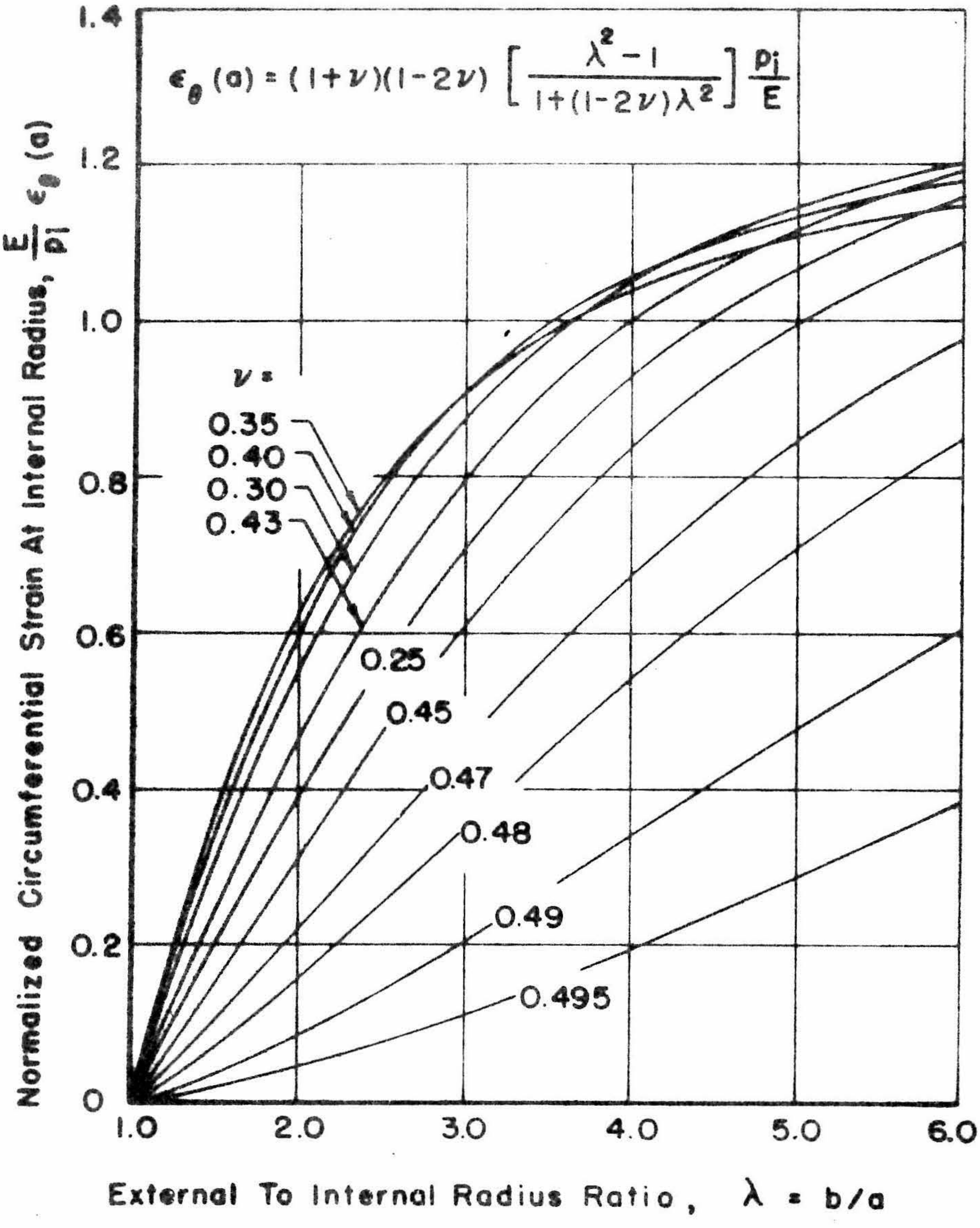


Fig. 2. Internal radial strain variation (Reference 8).

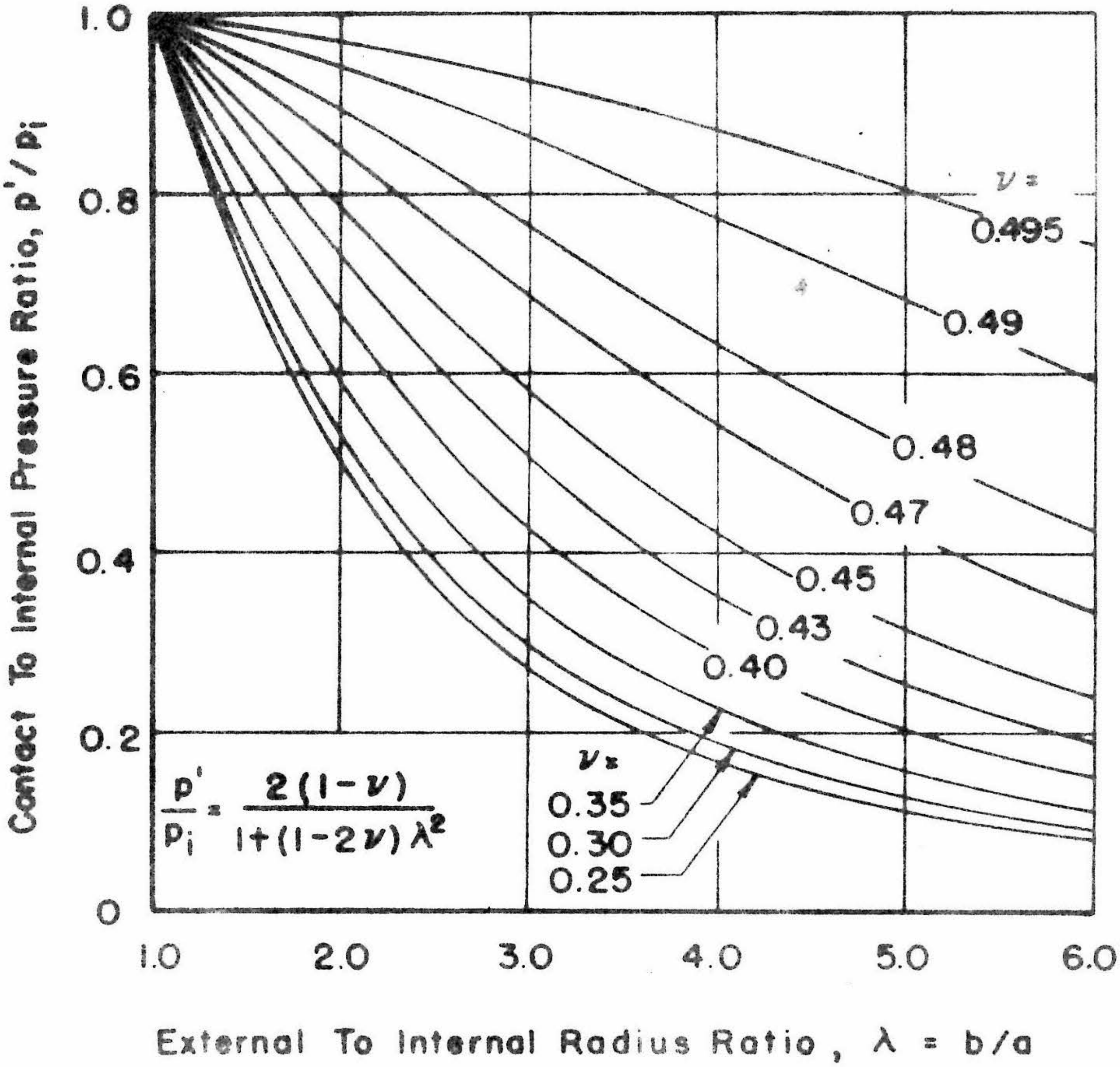


Fig. 3. Contact pressure variation (Reference 8).

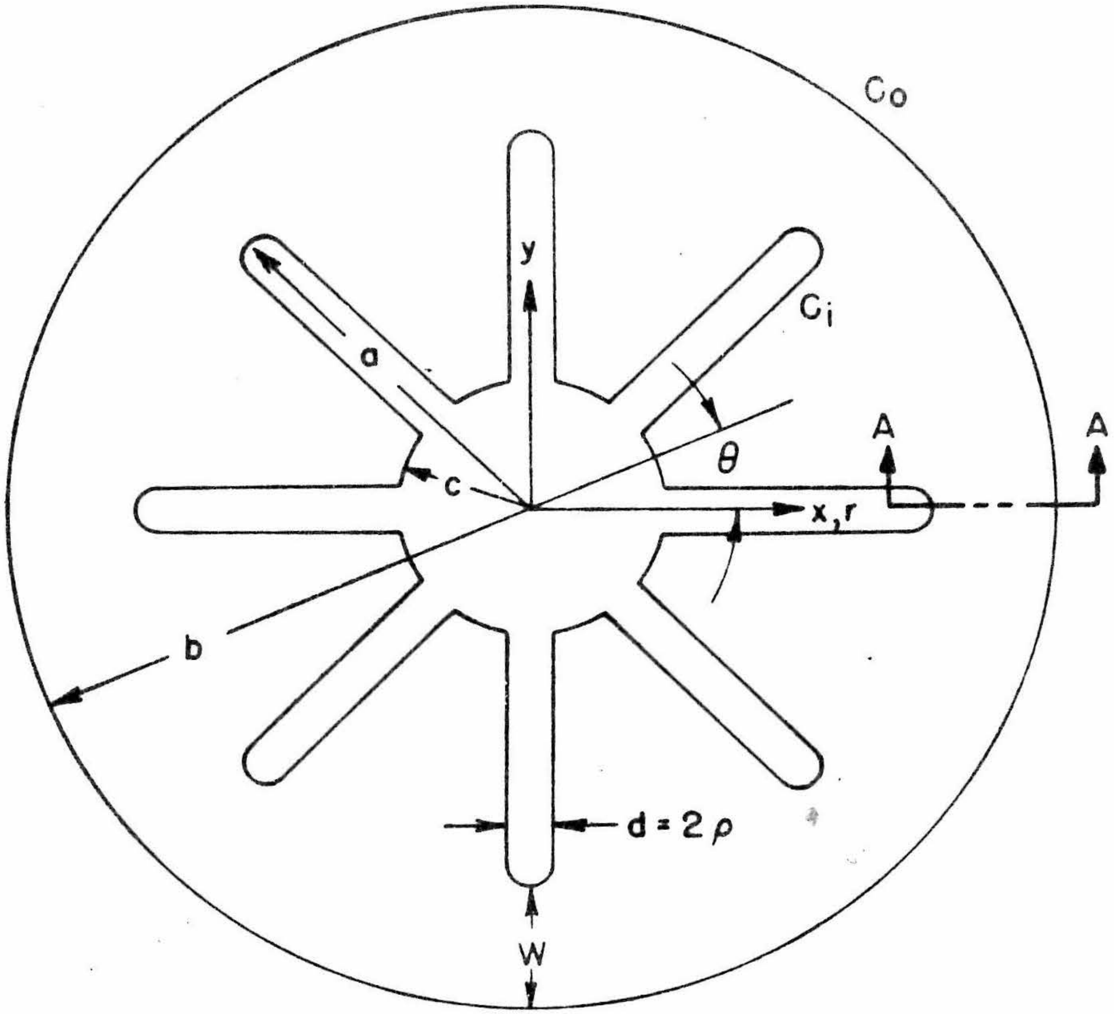


Fig. 4. Geometry of a slotted grain.

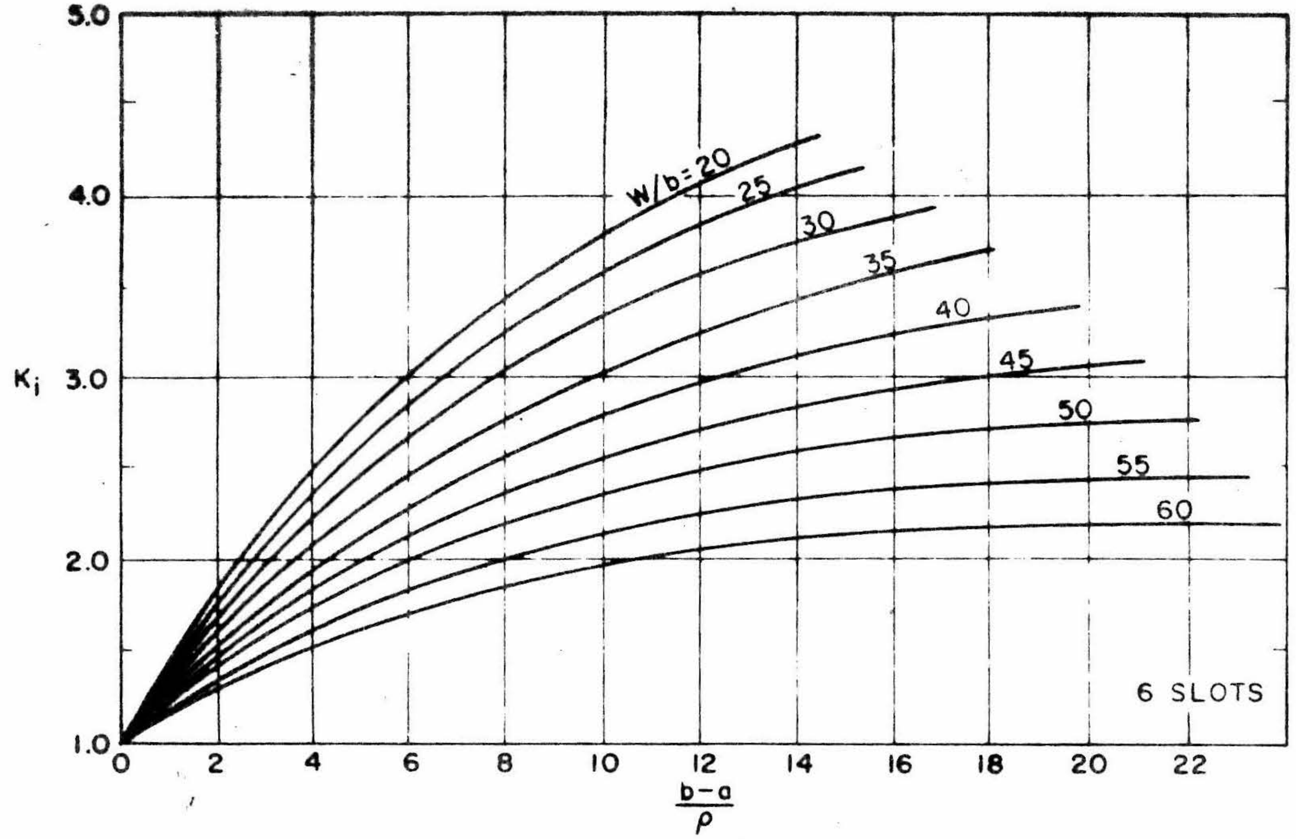


Fig. 5. Typical pressure stress concentration factor for a slotted grain (Reference 10).

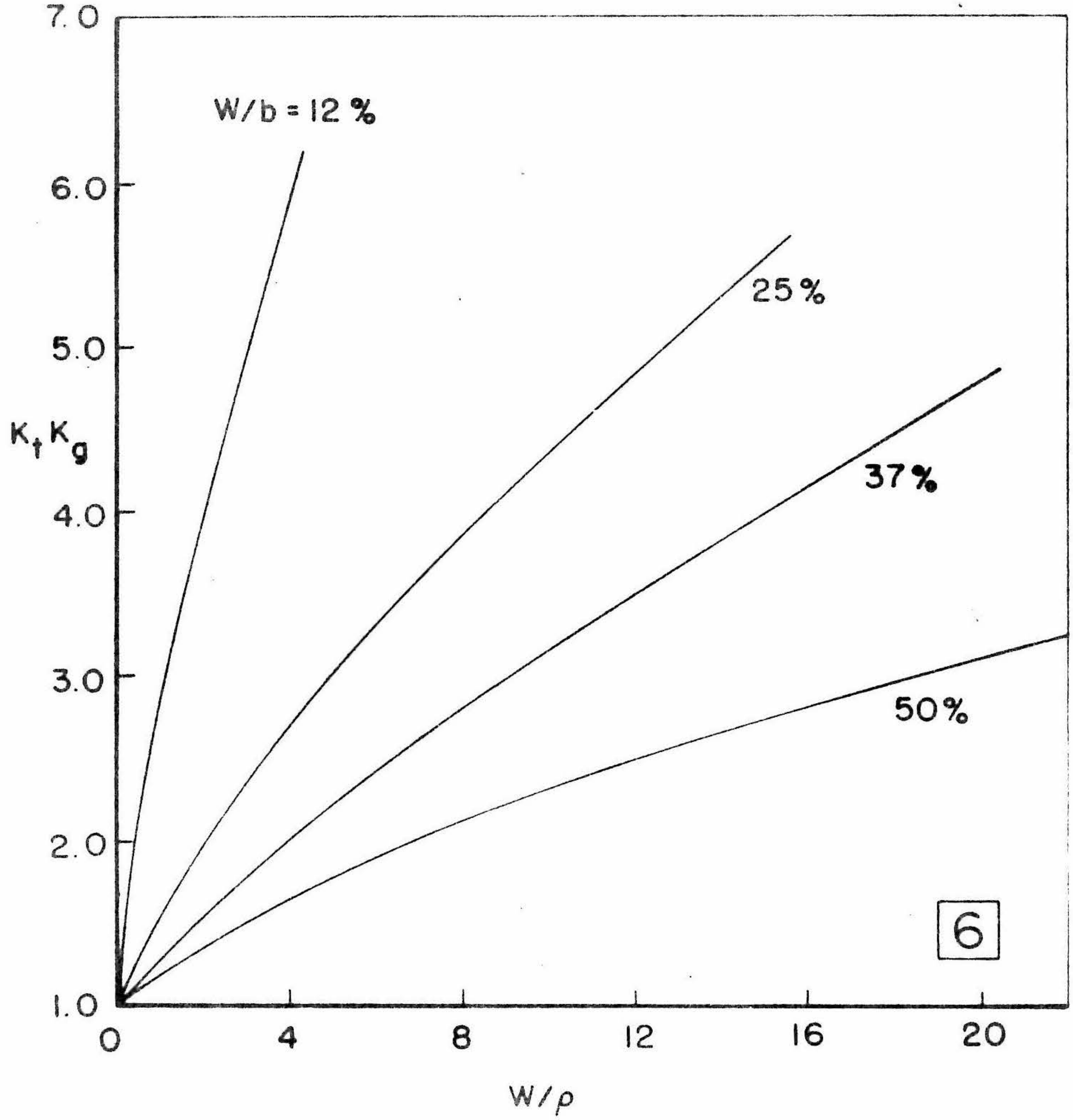
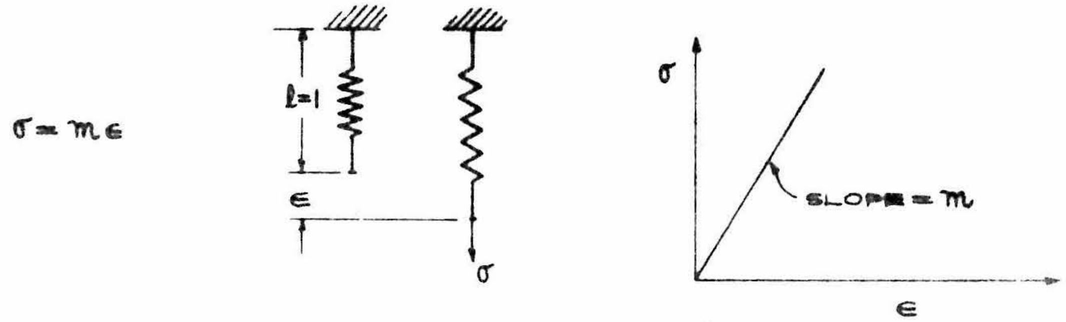


Fig. 6. Typical thermal stress concentration factor for a slotted grain (Reference 12).



7a. The Hooke single element model.

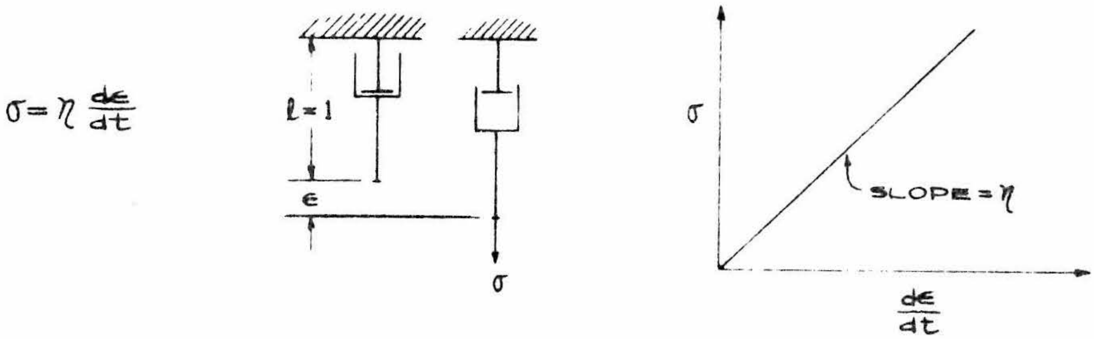


Fig. 7b. The Newtonian single element model.

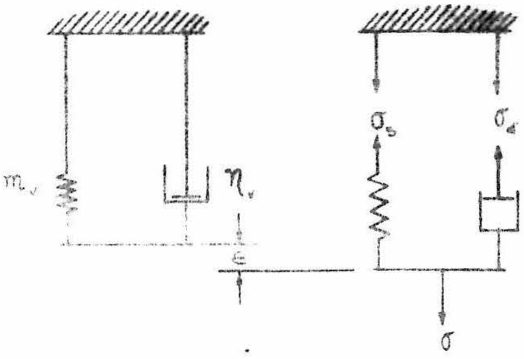
$\sigma = \sigma_d + \sigma_s$

WHERE $\quad \sigma_d = \eta_v \frac{d\epsilon}{dt}$

$\sigma_s = m_v \epsilon$

OPERATOR EQUATION:

$$\sigma(t) = \left(\eta_v \frac{d}{dt} + m_v\right)\epsilon(t)$$

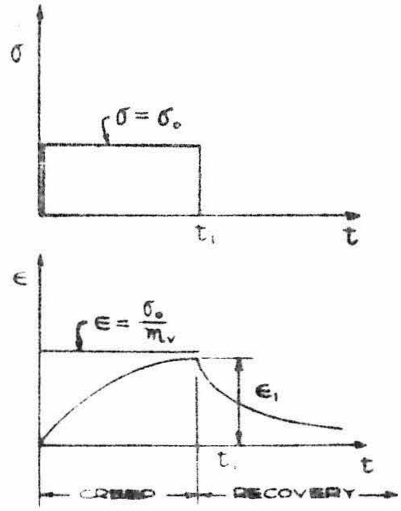


(a) MODEL

$\sigma = \sigma_o$; $\quad \epsilon(t) = \frac{\sigma_o}{m_v}\left(1 - e^{-\frac{t}{\tau_v}}\right)$: FOR $0 < t \leq t_1$

$\sigma = 0$; $\quad \epsilon(t) = \epsilon_1 e^{-\frac{(t-t_1)}{\tau_v}}$: FOR $t > t_1$

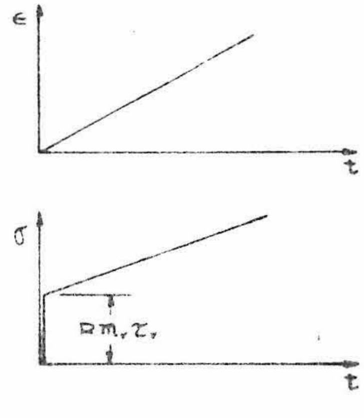
WHERE $\quad \tau_v = \frac{\eta_v}{m_v}$



(b) CREEP & RECOVERY

$\epsilon = Rt \quad$ (R = STRAIN RATE)

$\sigma = R m_v [t + \tau_v]$



(C) CONSTANT STRAIN RATE

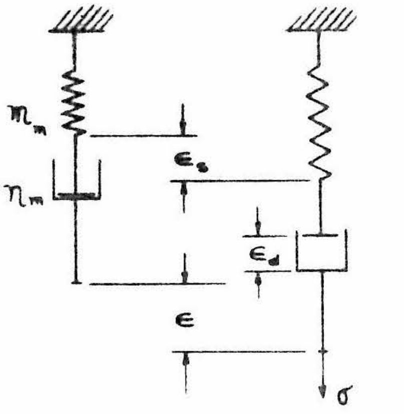
Fig. 7c. The Voigt double element model.

$$\epsilon = \epsilon_s + \epsilon_d$$

OR $$\frac{d\epsilon}{dt} = \frac{d\epsilon_s}{dt} + \frac{d\epsilon_d}{dt}$$

WHERE $$\frac{d\epsilon_s}{dt} = \frac{1}{m_m}\frac{d\sigma}{dt}$$

$$\frac{d\epsilon_d}{dt} = \frac{1}{\eta_m}\sigma$$



OPERATOR EQUATION:

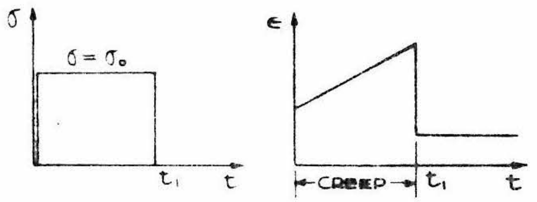
$$\left(\frac{1}{m_m}\frac{d}{dt} + \frac{1}{\eta_m}\right)\sigma(t) = \frac{d\epsilon(t)}{dt}$$

(a) MODEL

$\sigma = \sigma_0$; $\epsilon(t) = \frac{\sigma_0}{m_m}\left(1 + \frac{t}{\tau_m}\right)$: FOR $0 < t \le t_1$

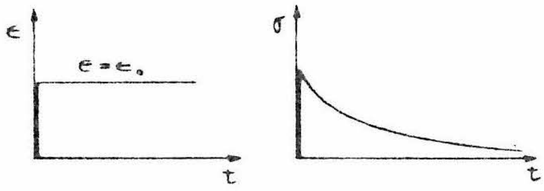
WHERE $\tau_m = \frac{\eta_m}{m_m}$

$\sigma = 0$; $\epsilon(t) = \frac{\sigma_0}{m_m}\frac{t_1}{\tau_m}$; FOR $t > t_1$



(b) CREEP

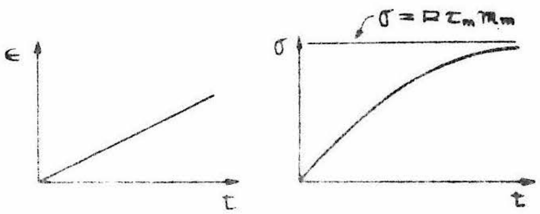
$\epsilon = \epsilon_0$; $\sigma(t) = \epsilon_0 m_m e^{-\frac{t}{\tau_m}}$: FOR $t > 0$



(c) RELAXATION

$\epsilon = Rt$ (R = STRAIN RATE)

$\sigma = R\tau_m m_m \left(1 - e^{-\frac{t}{\tau_m}}\right)$



(d) CONSTANT STRAIN RATE

Fig. 7d. The Maxwell double element model.

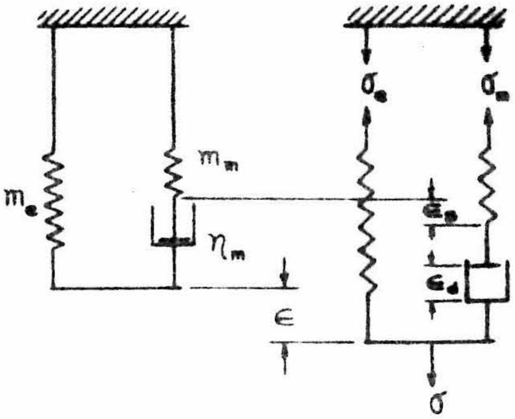
$$\epsilon = \epsilon_s + \epsilon_d$$

$$\sigma = \sigma_e + \sigma_m$$

WHERE $$\frac{d\epsilon_s}{dt} = \frac{1}{m_m}\frac{d\sigma_m}{dt}$$

$$\frac{d\epsilon_d}{dt} = \frac{\sigma_m}{\eta_m}$$

$$\epsilon = \frac{\sigma_e}{m_e}$$



OPERATOR EQUATION:

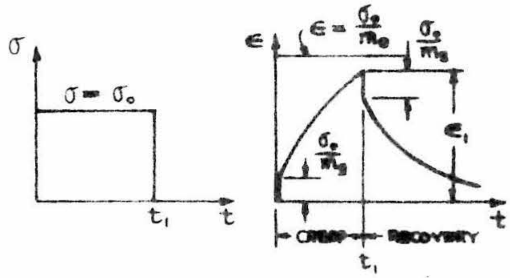
$$\sigma(t) = \left[m_e + \frac{m_m \frac{d}{dt}}{\left(\frac{d}{dt} + \frac{1}{\tau_m}\right)}\right]\epsilon(t) \qquad \text{OR} \qquad \sigma(t) = \frac{m_g\left(\frac{d}{dt} + \frac{m_e}{m_g}\frac{1}{\tau_m}\right)}{\left(\frac{d}{dt} + \frac{1}{\tau_m}\right)}\epsilon(t)$$

WHERE $\tau_m = \dfrac{\eta_m}{m_m}$; $m_m + m_e = m_g$

(a) MODEL

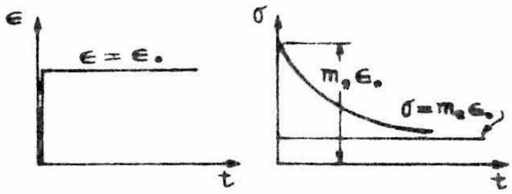
$$\epsilon = \left[1 - \left(1 - \frac{m_e}{m_g}\right)e^{-\frac{m_e}{m_g}\frac{t}{\tau_m}}\right]\frac{\sigma_o}{m_e} \; : \; \text{FOR } 0 < t \leq t_1$$

$$\epsilon = \left[\epsilon_1 - \frac{\sigma_o}{m_g}\right]e^{-\frac{m_e}{m_g}\frac{(t-t_1)}{\tau_m}} \; : \; \text{FOR } t > t_1$$



(b) CREEP & RECOVERY

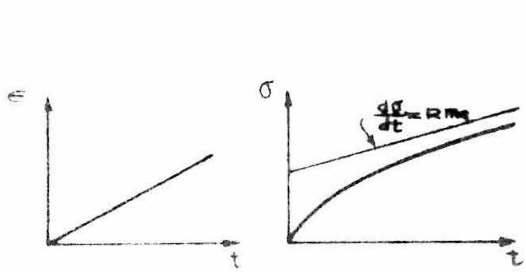
$$\sigma = \left[1 + \left(\frac{m_g}{m_e} - 1\right)e^{-\frac{t}{\tau_m}}\right]m_e\epsilon_o \; : \; \text{FOR } t > 0$$



(c) RELAXATION

$\epsilon = Rt$ (R = STRAIN RATE)

$$\sigma = \left[\frac{t}{\tau_m} + \left(\frac{m_g}{m_e} - 1\right)\left(1 - e^{-\frac{t}{\tau_m}}\right)\right]R m_e \tau_m$$



(d) CONSTANT STRAIN RATE

Fig. 7e. A typical three-element model.

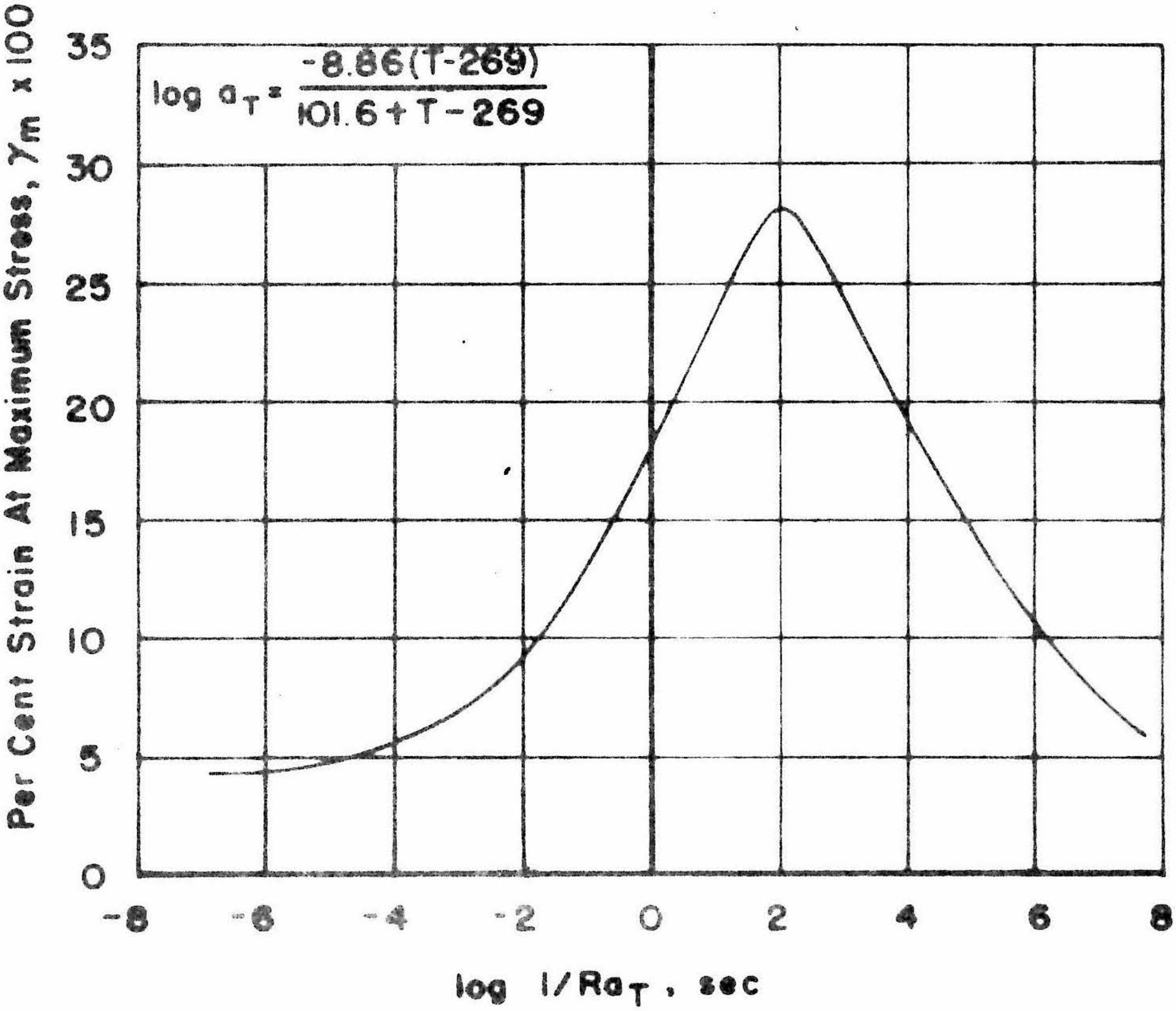


Fig. 8. Typical master curve (Reference 15).